Theorem 3.10 *Let $\mathcal{M}$ be the following system of axioms:*

$\emptyset \vdash X \to Y$ *if* $Y \subseteq X$ (trivial fds)	$(F1)$
$X \to Y \vdash X \to XY$ (fd-augmentation)	$(F2)$
$\{X \to Y, Y \to Z\} \vdash X \to Z$ (fd-transitivity)	$(F3)$
$X \twoheadrightarrow Y \vdash X \twoheadrightarrow \Omega - Y$ (mvd-complementation)	$(M1)$
$X \twoheadrightarrow Y \vdash WX \twoheadrightarrow VY$ *if* $V \subseteq W$ (mvd-augmentation)	$(M2)$
$\{X \twoheadrightarrow Y, Y \twoheadrightarrow Z\} \vdash X \twoheadrightarrow Z - Y$ (mvd-pseudotransitivity)	$(M3)$
$X \to Y \vdash X \twoheadrightarrow Y$ (mvds implied by fds)	$(FM1)$
$\{X \twoheadrightarrow Y, Y \to Z\} \vdash X \to Z - Y$ (mixed pseudotransitivity)	$(FM2)$

$\mathcal{M}$ is sound for the implication of fds and mvds.

Proof The soundness of axioms $F1$–$F3$ has been dealt with in Theorem 3.2 and the soundness of $M1$ and $FM1$ was shown in Corollary 3.1. It remains to show that axioms $M2$, $M3$ and $FM2$ are sound. Let PRS be a primitive relation scheme and let prs be a possible relation instance of PRS.

- *axiom $M2$:* Assume prs satisfies $X \twoheadrightarrow Y$. Let $t, u \in prs$ with $t[WX] = u[WX]$. We have to show that there exists $v \in sot$ such that $v[WXY] = t[WXY]$ and $v[XW(\Omega - VY)] = u[XW(\Omega - VY)]$. Since prs satisfies $X \twoheadrightarrow Y$ and since, in particular, $t[X] = u[X]$, it follows that there exists $v \in prs$ such that $v[XY] = t[XY]$ and $v[X(\Omega - Y)] = u[X(\Omega - Y)]$. Furthermore, since $t[W] = u[W]$ it follows that necessarily $v[W] = t[W] = u[W]$, whence the desired result.
- *axiom $M3$:* Assume prs satisfies both $X \twoheadrightarrow Y$ and $Y \twoheadrightarrow Z$. Let $t, u \in prs$ with $t[X] = u[X]$. We prove that there exists $w \in prs$ such that $w[X(Z-Y)] = t[X(Z-Y)]$ and $w[X(\Omega-(Z-Y))] = u[X(\Omega-(Z-Y))]$. For sake of clarity, note that $\Omega - (Z - Y)$ can be rewritten as $Y(\Omega - Z)$. Hence we can rewrite the last condition as $w[XY(\Omega - Z)] = u[XY(\Omega - Z)]$. Since prs satisfies $X \twoheadrightarrow Y$ (and hence, by Corollary 3.1, also $X \twoheadrightarrow \Omega - Y$) and since in particular $t[X] = u[X]$, it follows that there exists $v \in prs$ such that $v[X(\Omega - Y)] = t[X(\Omega - Y)]$ and $v[XY] = u[XY]$. In particular, $v[Y] = u[Y]$. Since prs satisfies $Y \twoheadrightarrow Z$ it then follows that there exists $w \in sot$ such that $w[YZ] = v[YZ]$ and $w[Y(\Omega - Z)] = u[Y(\Omega - Z)]$. Let us now fit everything together in order to show that w is the desired tuple. From $v[XY] = u[XY]$ and the construction of w it follows that $w[XY] = u[XY] = v[XY]$. Hence $w[XYZ] = v[XYZ]$ and $w[XY(\Omega - Z)] = u[XY(\Omega - Z)]$. Hence the second condition that w should satisfy is fulfilled. From $w[XYZ] = v[XYZ]$ and $v[X(\Omega - Y)] = t[X(\Omega - Y)]$ it follows that w and t agree on the intersection $XYZ \cap X(\Omega - Y) = X(Z - Y)$, and this is exactly the first condition that w must satisfy.
- *axiom $FM2$*: Assume prs satisfies $X \twoheadrightarrow Y$ and $Y \to Z$. Suppose t_1 and t_2 are tuples of prs satisfying $t_1[X] = t_2[X]$. Since prs satisfies $X \twoheadrightarrow Y$ there exists $u \in sot$ such that $u[XY] = t_1[XY]$ and $u[X(\Omega - Y)] = t_2[X(\Omega - Y)]$. Since $u[Y] = t_1[Y]$ and because of $Y \to Z$, it follows that $u[Z] = t_1[Z]$. From

this last equality and from $u[\Omega - Y] = t_2[\Omega - Y]$ it then follows that t_1 and t_2 agree on the intersection $Z \cap (\Omega - Y) = Z - Y$, i.e. $t_1[Z - Y] = t_2[Z - Y]$. Hence prs satisfies $X \rightarrow Z - Y$.

□

The next question that arises is of course whether the above axiom system is also complete. We invite the reader to try to infer the mvds and fds listed in Example 3.13 from the given set of mvds and fds using the rules of axiom system $\mathcal{M}$. You should come to the conclusion that this axiom system does indeed allow you to derive all these constraints from the given ones. Of course this is not a proof!

In order to shed some more light on this problem, let us re-examine the proof of Theorem 3.2 in which we showed the soundness, completeness and non-redundancy of an axiom system for fds. In the part where we showed the completeness, it turned out to be important to consider for a given set of fds and a given set of attributes X, all the fds implied by that set with X as left-hand side. Therefore, we needed to define $\overline{X}$. This suggests us to examine the set of all fds and mvds implied by a given set of fds and mvds with a fixed set of attributes as left-hand side. Is it possible to give a fairly simple description of this set? Therefore we first establish some additional inference rules.

Lemma 3.1 *The following rules can be derived from the axiom system $\mathcal{M}$ in Theorem 3.10. (and hence are sound):*

$$
\begin{array}{ll}
X \twoheadrightarrow Y \vdash X \twoheadrightarrow \Omega - Y \text{ (mvd-complementation)} & (M1) \\
\{X \twoheadrightarrow Y, X \twoheadrightarrow Z\} \vdash X \twoheadrightarrow Y \cap Z \text{ (mvd-intersection)} & (M4) \\
\{X \twoheadrightarrow Y, X \twoheadrightarrow Z\} \vdash X \twoheadrightarrow YZ \text{ (mvd-union)} & (M5) \\
\{X \twoheadrightarrow Y, X \twoheadrightarrow Z\} \vdash X \twoheadrightarrow Y - Z \text{ (mvd-difference)} & (M6)
\end{array}
$$

Proof First note that rule $M1$ is already an axiom of $\mathcal{M}$ that is only repeated here for sake of completeness. Let us consider rule $M4$. From $X \twoheadrightarrow Y$ one can deduce $X \twoheadrightarrow X(\Omega - Y)$ by applying first axiom $M1$ and then axiom $M2$. From $X \twoheadrightarrow Z$ one can derive $X(\Omega - Y) \twoheadrightarrow Z$, again by using axiom $M2$. If we now use the pseudotransitivity axiom $M3$ on $X \twoheadrightarrow X(\Omega - Y)$ and $X(\Omega - Y) \twoheadrightarrow Z$ we get $X \twoheadrightarrow Z - (X(\Omega - Y))$. Since $Z - (X(\Omega - Y))$ equals $(Y \cap Z) - X$, a final application of axiom $M2$ yields the desired result. Rules $M5$ and $M6$ can be easily derived from axiom $M1$ and rule $M4$ knowing that $YZ = \Omega - ((\Omega - Y) \cap (\Omega - Z))$ and $Y - Z = Y \cap (\Omega - Z)$ and are left as an exercise to the reader.

□

A set of sets that is closed under complementation and intersection (and, as a consequence, under union and set difference) can be described as consisting of all possible unions of members of the partition induced by that set. We are going to use this idea in order to describe the set of fds and mvds that can be derived from a given set of fds and mvds.

Theorem 3.11 *Let $PRS = (\Omega, \Delta, dom)$ be a primitive relation scheme and let $\mathcal{FD}$ and $\mathcal{MD}$ be the sets of all fds and mvds of PRS respectively. Let $SC \subseteq \mathcal{FD} \cup \mathcal{MD}$. Let $DepB(X)$ be the partition induced by $\{Y \mid X \rightarrow\!\!\!\rightarrow Y \in SC^+_{\mathcal{FD}\cup\mathcal{MD}}\}$[3] and let $\overline{X} = \{A \mid X \rightarrow A \in SC^+_{\mathcal{FD}\cup\mathcal{MD}}\}$.[4] Then:*

- *$X \rightarrow\!\!\!\rightarrow Y \in SC^+$ if and only if there exists $\mathcal{Y} \subseteq DepB(X)$ such that $Y = \bigcup \mathcal{Y}$;*
- *$X \rightarrow Y \in SC^+$ if and only if $Y \subseteq \overline{X}$;*
- *if $A \in \overline{X}$, then $\{A\} \in DepB(X)$.*

The set $DepB(X)$ is called the dependency basis of X for the set SC.

Proof Left as an exercise to the reader.

□

Example 3.14 Let us consider a primitive relation scheme PRS with:

- $\Omega = \{A, B, C, D, E, F, G\}$
- $SC = \{AB \rightarrow\!\!\!\rightarrow CDE, AB \rightarrow\!\!\!\rightarrow EFG\}$.

Let $X = AB$. Since SC does not contain fds, the rules of Theorem 3.10 allow only to derive trivial fds. Hence $\overline{X} = \{A, B\}$ and $\{A\}$ and $\{B\}$ certainly belong to $DepB(X)$. By rules $M1$, $M4$ and $M6$, $AB \rightarrow\!\!\!\rightarrow CD$, $AB \rightarrow\!\!\!\rightarrow E$ and $AB \rightarrow\!\!\!\rightarrow FG$ can also be derived from SC. The reader can check by constructing a counterexample that it is impossible to infer $AB \rightarrow\!\!\!\rightarrow C$, $AB \rightarrow\!\!\!\rightarrow D$, $AB \rightarrow\!\!\!\rightarrow F$ or $AB \rightarrow\!\!\!\rightarrow G$ from SC. Since the rules in Theorem 3.10 are sound, this mvds are not in SC^+ either. Hence $DepB(X) = \{A, B, CD, E, FG\}$. Recall that, in accordance with an earlier remark, A, B and E stand for the sets $\{A\}$, $\{B\}$ and $\{C\}$ respectively.

□

Of course, a more efficient procedure to compute $\overline{X}$ and $DepB(X)$ is needed. The algorithm we give here is based on [15].

Algorithm 3.3 *Attributeset Closure and Dependency Basis*

Input: $X \subseteq \Omega$ and SC, a set of fds and mvds of a primitive relation scheme $PRS = (\Omega, \Delta, dom)$.

Output: $\overline{X}$, $DepB(X)$

Method:

```
var  OLDX, NEWX, XPLUS, DBU, DBV, W : set of attributes;
     OLDD, NEWD, DEPBX : set of sets of attributes;
NEWX := X;
NEWD := {{A} | A ∈ X} ∪ {Ω − X};
repeat
    OLDX := NEWX;
    OLDD := NEWD;
```

[3]In the remainder of this section, we shall write $SC^+_{\mathcal{FD}\cup\mathcal{MD}}$ as SC^+ for short.

[4]If we suppose that $SC \subseteq \mathcal{FD}$, we get back the definition of $\overline{X}$ given in the proof of Theorem 3.2.

for each $U \rightarrow V$ **in** SC **do**
  $DBU := \bigcup\{W \mid W \in NEWD \; \& \; W \cap U \neq \emptyset\}$;
  $DBV := V - DBU$;
  if $DBV \neq \emptyset$
   then
    begin
     $NEWX := NEWX \cup DBV$;
     $NEWD := \{W - DBV \mid W \in NEWD \; \& \; W - DBV \neq \emptyset\}$
      $\cup\{\{A\} \mid A \in DBV\}$
    end
od
for each $U \rightarrow\!\!\rightarrow V$ **in** SC **do**
  $DBU := \bigcup\{W \mid W \in NEWD \; \& \; W \cap U \neq \emptyset\}$;
  $DBV := V - DBU$;
  if $DBV \neq \emptyset$
   then
    for each W **in** $NEWD$ **do**
     if $(W \cap DBV \neq \emptyset)$ **and** $(W \cap DBV \neq W)$
      then
       $NEWD := (NEWD - \{W\})$
        $\cup\{W \cap DBV, W - DBV\}$;
    od
od
until $(NEWX = OLDX)$ **and** $(NEWD = OLDD)$;
$XPLUS := NEWX$;
$DEPBX := NEWD$;
return$(XPLUS, DEPBX)$

□

Theorem 3.12 *Algorithm 3.3 is correct and computes attributeset closure and dependency basis in polynomial time.*

Proof We shall only give an outline of the proof. The reader is invited to fill out the details. First, we have to show that the operations performed on $NEWX$ and $OLDX$ do not violate the following conditions which are trivially satisfied after initialization:

- $X \rightarrow NEWX \in SC^+$;
- for all $W \in NEWD$, $X \rightarrow\!\!\rightarrow W \in SC^+$.

This can be easily achieved using various axioms and the rules we derived from them. Then we have to show that $X \rightarrow\!\!\rightarrow W'$ is *not* in SC^+ for any proper subset W' of a set W in $DEPBX$. This can be done by showing that from the sets

$$\{X \to Y \mid Y \subseteq XPLUS\}$$

and

$$\{X \to\!\!\!\to Y \mid Y \textit{ is a union of some members of } DEPBX\}$$

no other fds and mvds can be derived using an axiom of $\mathcal{M}$ in Theorem 3.10. Finally, the time complexity of Algorithm 3.3 can be computed in a straightforward manner.

□

It is still possible to improve the time complexity of Algorithm 3.3. There exist various quadratic and even almost linear algorithms in the literature ([53, 59, 96]). We do not intend however to discuss them here. We now illustrate Algorithm 3.3 with an example.

Example 3.15 Let us consider a primitive relation scheme PRS with:

- $\Omega = \{A, B, C, D, E, F, G\}$
- $SC = \{AB \to\!\!\!\to CD, C \to\!\!\!\to F, C \to E\}$.

and calculate the dependency basis of $X = AB$ using Algorithm 3.3. Initially we have:

$$NEWX{=}AB$$
$$NEWD{=}\{A, B, CDEFG\}$$

After the application of $AB \to\!\!\!\to CD$ we get:

$$NEWX{=}AB$$
$$NEWD{=}\{A, B, CD, EFG\}$$

An application of $C \to\!\!\!\to F$ gives:

$$NEWX{=}AB$$
$$NEWD{=}\{A, B, CD, EG, F\}$$

Finally, after the use of $C \to E$ we get:

$$NEWX{=}ABE$$
$$NEWD{=}\{A, B, CD, E, F, G\}$$

It is easily seen that another pass through SC does not lead to any additional changes. Hence the algorithm gives:

$$\overline{X}{=}XPLUS = ABE$$
$$DepB(X){=}DEPBX = \{A, B, CD, E, F, G\}$$

□

In order for this algorithm to be the basis of an algorithm to decide the implication problem for fds and mvds, we have to prove the completeness of the axiom system $\mathcal{M}$ introduced in Theorem 3.10, for which we are now ready.

Theorem 3.13 *The axiom system $\mathcal{M}$ in Theorem 3.10 is sound, complete and non-redundant for the implication of fds and mvds.*

Proof First recall from Theorem 3.10 that $\mathcal{M}$ is sound. Let PRS be a primitive relation scheme. Let $\mathcal{FD}$ be the set of all fds of PRS and let $\mathcal{MD}$ be the set of all mvds of PRS. Let $SC \subseteq \mathcal{FD} \cup \mathcal{MD}$. We have to show that $SC^* \subseteq SC^+$.[5] Let $W_1, \ldots, W_k$ be those member of $DepB(X)$ that are not contained in $\overline{X}$. We now construct the following relation s:

$\overline{X}$	W_1	W_2	$\cdots$	W_k
$0\ldots0$	$0\ldots0$	$0\ldots0$	$\cdots$	$0\ldots0$
$0\ldots0$	$1\ldots1$	$0\ldots0$	$\cdots$	$0\ldots0$
$0\ldots0$	$0\ldots0$	$1\ldots1$	$\cdots$	$0\ldots0$
$0\ldots0$	$1\ldots1$	$1\ldots1$	$\cdots$	$0\ldots0$
		$\vdots$		
$0\ldots0$	$1\ldots1$	$1\ldots1$	$\cdots$	$1\ldots1$

So s contains 2^k tuples.[6] We now show that s satisfies all the fds and mvds of SC. Therefore, let $U \to V \in SC$. Let W be the union of those W_i's that intersect U. (W may be empty). Clearly, $\overline{X}W \to V \in SC^+$. Now let t_1 and t_2 be tuples of s such that $t_1[U] = t_2[U]$. Note that by construction of s it follows that $t_1[\overline{X}W] = t_2[\overline{X}W]$. By Theorem 3.11, we have that $X \twoheadrightarrow \overline{X}W$ is in SC^+. Hence by mixed pseudo-transitivity, $X \to V - \overline{X}W$ is in SC^+, whence $V - \overline{X}W \subseteq \overline{X}$. By construction of s it then follows that $t_1[V - \overline{X}W] = t_2[V - \overline{X}W]$. Since we already know that t_1 and t_2 agree on $\overline{X}W$, we get that $t_1[V] = t_2[V]$ whence satisfaction of $U \to V$ by s. Now assume that $U \twoheadrightarrow V$ is in SC. We must show that whenever there exist tuples t_1 and t_2 that agree on U, there also exists a tuple t such that $t[UV] = t_1[UV]$ and $t[U(\Omega - V)] = t_2[U(\Omega - V)]$. Let W be again the union of those W_i's that intersect U. Then it follows from the construction of s that $t_1[\overline{X}W] = t_2[\overline{X}W]$. Also, by Theorem 3.11, it follows that $X \twoheadrightarrow \overline{X}W$ is in SC^+. By mvd-augmentation, $\overline{X}W \twoheadrightarrow V$ is in SC^+. Hence, by mvd-transitivity, $X \twoheadrightarrow V - \overline{X}W$ is in SC^+. So $V - \overline{X}W$ is a union of W_i's. From the construction of s the existence of the above described tuple t now easily follows.

Now suppose that $X \to Y$ is in SC^*. Since s satisfies all the dependencies in SC, it also satisfies all those of SC^*. Hence $Y \subseteq \overline{X}$ by construction of s, which in turn implies that $X \to Y$ is in SC^+. Similarly, suppose that $X \twoheadrightarrow Y$ is in SC^*. Then again s must satisfy this mvd and this can only be the case if Y is the union of some members of $\overline{X}$, whence $X \twoheadrightarrow Y \in SC^+$. Hence $SC^* \subseteq SC^+$ as had to be shown.

It only remains to be shown that the axiom system is non-redundant. This can be done according to the principle used in the proof of Theorem 3.2 in a straightforward way. Therefore we leave this part of the proof to the reader. $\square$

[5] SC^* of course denotes $SC^*_{\mathcal{FD}\cup\mathcal{MD}}$.

[6] Note that in case only fds are involved, $k = 1$ and $W_1 = \Omega - \overline{X}$. Hence the relation instance s constructed above then becomes the relation instance r constructed in the proof of Theorem 3.2.

From Theorem 3.12 and Theorem 3.13 it immediately follows:

Corollary 3.2 *The implication problem for fds and mvds is decidable in polynomial time.*

□

3.4 Join Dependencies

In the previous section, we presented mvds as a necessary and sufficient condition to decompose a relation into two subrelations without losing information. We shall however not end our discussion on decomposition-related constraints here, since there exist situations, as was shown by J.-M. Nicolas [81], in which a relation can be decomposed into three subrelations but not into two. We illustrate this point with an example.

Example 3.16 Consider again the relation scheme $RS = (\Omega, \Delta, dom, M, SC)$ of Example 3.12. Recall in particular that SC consists of only one constraint saying that whenever a *DRINKER* drinks a *BEER*, he drinks that *BEER* in every *BAR* where it is served. We showed that this constraint can be represented by the mvd $BEER \to\to DRINKER$ (or, equivalently, by $BEER \to\to BAR$).

In this example, we consider a relation scheme RS' obtained from RS by slightly modifying the only constraint. We now assume that *whenever a DRINKER drinks a BEER and whenever that DRINKER frequents a BAR in which that BEER is served, he drinks that BEER in that BAR*. We call this constraint sc'. Let us now consider the following instance of RS':

DRINKER	*BEER*	*BAR*
Jones	*Tuborg*	*Tivoli*
Jones	*Tuborg*	*Far West*
Jones	*Carlsberg*	*Tivoli*
Smith	*Tuborg*	*Tivoli*

It is readily verified that this instance satisfies the new constraint sc'. It is also easily seen that none of the mvds

$$\begin{aligned} BEER &\to\to DRINKER \\ DRINKER &\to\to BAR \\ BAR &\to\to BEER \end{aligned}$$

holds. Hence it is not possible to decompose RS' into two subschemes without losing information. It is however easy to see that there exists a lossless decomposition of RS' into *three* subschemes, namely the projections of RS' onto $\{DRINKER, BEER\}$, $\{BEER, BAR\}$ and $\{DRINKER, BAR\}$ respectively. If we apply this decomposition strategy on the above instance, we get:

DRINKER	*BEER*
Jones	*Tuborg*
Jones	*Carlsberg*
Smith	*Tuborg*

BEER	*BAR*
Tuborg	*Tivoli*
Tuborg	*Far West*
Carlsberg	*Tivoli*

DRINKER	*BAR*
Jones	*Tivoli*
Jones	*Far West*
Smith	*Tivoli*

It is easily seen that we can recover the original instance by performing a natural join on these projections. Moreover, a closer examination of the constraint sc' reveals that it actually *says* that RS' can be decomposed into the three subschemes mentioned above.

□

Nicolas [81] called a constraint such as the one we introduced in the above example, which yields a necessary and sufficient condition for a relation to be decomposable into three subrelations, a *mutual dependency*. The generalization is of course obvious [91]:

Definition 3.9 Let $PRS = (\Omega, \Delta, dom)$ be a primitive relation scheme. Let $X_1, \ldots, X_k \subseteq \Omega$ with $\bigcup_{i=1}^{k} X_i = \Omega$. A *join dependency* $X_1 \bowtie \cdots \bowtie X_k$ over PRS is a constraint that is satisfied by a possible relation instance prs if and only if for all $t_1, \ldots, t_k \in prs$ with $t_i[X_i \cap X_j] = t_j[X_i \cap X_j]$ for all $i, j = 1, \ldots, k$ there exists a tuple $t \in prs$ such that $t[X_i] = t_i[X_i]$ for all $i = 1, \ldots, k$.

□

Hence the constraint sc' in Example 3.16 is a join dependency (jd) with three components that can be denoted as

$$\{DRINKER, BEER\} \bowtie \{BEER, BAR\} \bowtie \{DRINKER, BAR\}.$$

From Definition 3.9 we can immediately derive:

Theorem 3.14 *Let $RS = (\Omega, \Delta, dom, M, SC)$ be a relation scheme. Let $X_1, \ldots, X_k \subseteq \Omega$ with $\bigcup_{i=1}^{k} X_i = \Omega$. $SC \models X_1 \bowtie \cdots \bowtie X_k$ if and only if for each relation instance r of RS we have that $r = \Pi(r, X_1) \bowtie \cdots \bowtie \Pi(r, X_k)$.*

□

Theorem 3.14 explains the notation we used to denote a jd. Theorem 3.14 also yields the following corollary:

Corollary 3.3 *Let $PRS = (\Omega, \Delta, dom)$ be a primitive relation scheme.*

- *Let $X, Y \subseteq \Omega$. Then $X \twoheadrightarrow Y \Leftrightarrow XY \bowtie X(\Omega - Y)$.*
- *Let $X_1, X_2 \subseteq \Omega$ with $X_1 \cup X_2 = \Omega$. Then $X_1 \bowtie X_2 \Leftrightarrow X_1 \cap X_2 \twoheadrightarrow X_1$.*[7]

Proof Follows immediately from Theorem 3.14 and Theorem 3.9.

□

[7] Instead of X_1 we might also have written X_2, $X_1 - X_2$ or $X_2 - X_1$.